Chapter 5
Arithmetic Properties

Natural numbers (1,2,3,4….) are the building blocks from which a large part of mathematics is constructed. With zero also recognised as a number we have the set of whole numbers. To construct mathematics numbers do not work in isolation but in combinations. Mathematics is constructed by the patterns created by the combinations and the laws these operations obey.

The Property of Closure

The set of whole numbers is closed for addition and multiplication. These operations performed on whole numbers always result in a whole number.

The set of whole numbers is not closed for subtraction or division. These operations when performed on whole numbers may result in the need for numbers from outside the set of whole numbers.

Addition and multiplication of whole numbers will always result in a whole number, which is a reason why students find these operations the easiest to understand. Students are more likely to have an understanding of whole numbers.

A limited range of subtractions with whole numbers result in a whole number answer. It is very common for students to be taught that you can only subtract a smaller whole number from a larger whole number. However it is possible to subtract a larger number from a smaller number if the set of numbers is expanded to include positive and negative integers. In the absence of knowledge of negative integers when faced with the expression 3 – 7 students will either say it cannot be done, give an answer of zero, or they attempt to make sense of the expression and see 7 – 3.

A limited range of divisions with whole numbers result in a whole number answer. Many students are only presented with divisions giving whole number answers which in turn limits their understanding of division as an operation. Divisions 8 ÷ 3 can be restricted to the use of natural numbers by leaving a whole number remainder. However to deal with divisions like 3 ÷ 8, and to get an exact solution to 8 ÷ 3 requires the set of numbers to be expanded to include rational numbers - numbers that can be expressed as fractions.

For further exploration of numbers refer to other Teacher Handbooks in this series:
Numbers and The Number System.
Fractions, Decimals & Percentages.

The Property of Commutativity

The operations of addition and multiplication are commutative - the order in which the operation is performed makes no difference to the answer.

For any numbers a and b

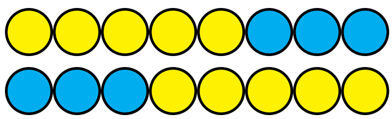
a + b = b + a

a x b = b x a

Subtraction and division are not commutative; the order in which the operation is performed is of vital importance. A change of order will result in different answers.

Exploring the commutative property for addition needs to be made explicit before students begin to memorise basic addition facts. For each basic fact they learn they are actually learning two facts.

5 + 3 = 3 + 5



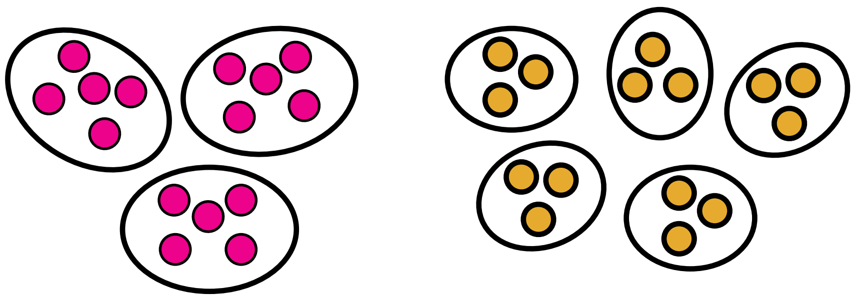
Once students understand the commutative property for addition they can use counting on from the largest number for efficiency when solving addition problems by counting, irrespective of which number is presented first in an equation.
The commutative property is actually more interesting when it is absent. In subtraction the order of subtraction makes a lot of difference.
Subtraction is not commutative: When working with just the natural numbers
7 – 4 = 3 but 4 – 7 does not have an answer. You will need to work with positive and negative integers to get an answer, but even then the answers are certainly not equal.

7 – 4 = 3   4 – 7 = - 3

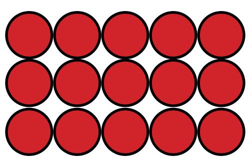
The order of the numbers in a subtraction is vital, but students often over generalise the commutative property and will even read problems written as 4 - 7 as “seven minus four”.
It is much more convenient to perform an operation when the numbers can be arranged in whatever order suits us best. Some mistakes

students make in subtraction are due to their failure to understand that subtraction is not commutative.
In problems like 63 – 37 a very common answer students give is 34
6 – 3 (60 – 30) and 7 – 3 = 4 instead of 3 – 7 = -4
All too often they have been drilled to believe “you cannot take a bigger number away from a smaller number”. Therefore for convenience and in over generalising the commutative property, they swap numbers around without understanding subtraction is not commutative.

Students need plenty of experience of the commutative law, for multiplication, for it to become part of their mathematical understanding and therefore halve the number of multiplication facts to learn. They need to be able to use commutativity spontaneously when solving problems.
Students are generally introduced to multiplication as repeated addition
3 x 5 meaning 3 groups of 5 or 5 + 5 + 5
5 x 3 meaning 5 groups of 3 or 3 + 3 + 3 + 3 + 3
These two situations look very different when illustrated using an equal grouping model.

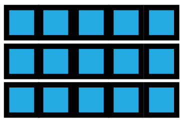


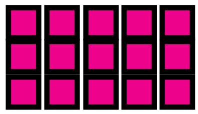
However using an array model to illustrate shows how 3 x 5 gives the same result as 5 x 3.



The array model illustrates the commutative property of multiplication by reading it horizontally for one equation and vertically for the other equation.

Many students appear to have initial difficulty in reading an array both horizontally and vertically at the same time, without physically turning the array through ninety degrees or moving themselves through ninety degrees.
A visual model to assist students to transition from an equal grouping, repeated addition model to an array model for multiplication is by using number strips. (available from The Wilkie Way members content area numeracy resources at www.thewilkieway.co.nz)





These models can be physically overlaid on top of each other.

The word "times" for the multiplication symbol requires an understanding of commutativity.
5 (times 3) is 5 + 5 + 5
(5 times) 3 is 3 + 3 + 3 + 3 + 3
Likewise "3 times as much" or "multiplied by 3" is a relationship between quantities. Each quantity is increased so it becomes three times the original size. Every number becomes 3 times the original number.
When an understanding of commutativity has been achieved a student can use 3 x 5 or 5 x 3 as an expression of all the interchangeable ideas:
a) three groups of five
b) five groups of three
c) three times (as much as) five
d) five times (as much as) three
e) three multiplied by five (three increased to five times its size)
f) five multiplied by three (five increased to three times its size)

Division is not commutative:
12 ÷ 3 = 4 3 ÷ 12 = 1/4
The results are different and require the use of rational numbers (fractions).
The division symbol represents a range of mathematical concepts, which is why students think division is hard and often fail to make all the connections required to fully understand the symbol. (See chapter 4 Signs and symbols).

The Property of Association

The operations of addition and multiplication obey the associative law – when adding or multiplying three or more whole numbers, the order in which they are added or multiplied does not affect the answer.
For any numbers a, b and c
$(a + b) + c = a + (b + c) = (a + c) + b$
$(a \times b) \times c = a \times (b \times c) = (a \times c) \times b$

When a student is counting from one to solve an addition, adding three numbers is no more complicated than adding two numbers. However, when students have recall of some basic addition facts they can regroup the numbers to make the addition easier.

4 + 8 + 6 =

If the student has recall of facts to ten and the teens numbers then
4 + 6 = 10 and 10 + 8 = 18

The associative property is used when students perform mental additions like 25 + 7
= 20 + (5 + 7)
= 20 + 12
= (20 + 10) + 2
= 32

Or they may use an alternative strategy, still making use of the associative property:
25 + 7
= 25 + (5 + 2)
= (25 + 5) + 2
= 30 + 2
= 32

A student who does not understand the regrouping, (part/wholing) implied by the associative property of addition cannot make use of basic fact knowledge and is therefore limited to counting on 7 from the 25.

When multiplying three numbers together the order in which the multiplication is performed can be varied for convenience.

3 x 4 x 5 can be solved	(3 x 4) x 5	3 x (4 x 5)	(3 x 5) x 4
	= 12 x 5	= 3 x 20	= 15 x 4
	= 60	= 60	= 60

The associative property is used when multiplying large numbers using place value knowledge.

40 x 7
= (4 x 10) x 7
= (4 x 7) x 10
= 28 x 10
= 280

The associative property is also used when using a doubling and halving strategy in multiplication.

3 x 16
= 3 x (2 x 8)
= (3 x 2) x 8
= 6 x 8
= 48

Many students assume that the commutative property and associative property are interchangeable and may miss the difference between them. Confusion often arises as we use both the properties simultaneously.
For example 4 + 8 + 6 + 5

The associative property is used to reorder the numbers:
(4 + 6) + (8 + 5)
10 + 13

The commutative property is used to add 13 + 10 = 23

Subtraction is not associative:
If you change the order in which the operations are performed different answers will result.

9 – 5 – 3	(9 – 5) – 3	9 – (5 – 3)
	= 4 – 3	= 9 – 2
	= 1	= 7

Division is not associative:
If you change the order in which the operations are performed different answers will result.

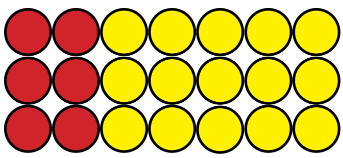
24 ÷ 3 ÷ 4	(24 ÷ 3) ÷ 4	24 ÷ (3 ÷ 4)
	= 8 ÷ 4	= 24 ÷ 3/4
	= 2	= 32

The Distributive Property of Multiplication over Addition

For any numbers a,b and c
a x (b + c) = (a x b) + (a x c)
The sum of the addends can be multiplied by the factor or each addend can be multiplied by the factor and then the sum found.

Students learning multiplication facts can make use of this property to use the multiplication facts they can recall to strategise unknown facts. Students who have recall of the five times and two times tables can use the distributive property to work out the seven times table.

This is best illustrated using an array model of multiplication.

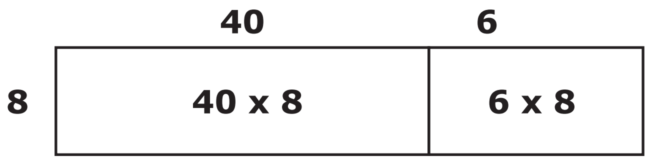


3 x 7
3 x (2 + 5) = (3 x 2) + (3 x 5)
= 6 + 15
= 21

Using recall of square numbers and the distributive property allows students another strategy to recall multiplication facts quickly.

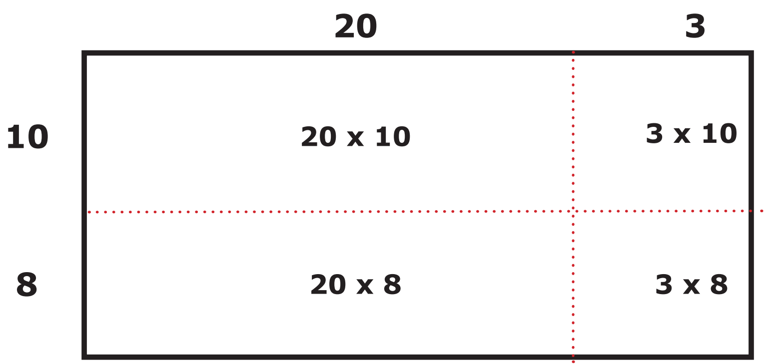
6 x 7 = (6 x 6) + (6 x 1)
= 36 + 6 (using the associative property 30 + 6 + 6)
= 42

This distributive property is used for multiplying multi digit numbers using place value distribution. An understanding of area allows a rectangle or an empty array to provide a visual representation of the distributive property.



46 x 8	=	(40 x 8) + (6 x 8)
40 x 8	=	320 (using the associative property 4 x 8 x 10)
6 x 8	=	48
320 + 48	=	368

An empty array provides a visual model of the distributive property of multiplication as applied to multi digit by multi digit multiplications which clearly shows the full distribution which will enable students to make sense of a standard written method for multi digit multiplication.



23 x 18 = (20 x 10) + (20 x 8) + (3 x 10) + (3 x 8)

20 x 10 = 200 (using the associaive property or place value knowledge
2 x 10 x 10)

20 x 8 = 160 (2 x 8 x 10)

3 x 10 = 30

3 x 8 = 24

200 + 160 + 30 + 24 = 414

Special Numbers in Arithmetic - 0 and 1

Zero (0) is the Additive Identity.

It is the only number which does not change the value of any number to which it is added.

The additive identity property can be expressed in four different ways and each way can be written in two equivalent forms.

n + 0 = n or n = n + 0
0 + n = n or n = 0 + n
n - 0 = n or n = n - 0
n - n = 0 or 0 = n - n

The additive property seems to be intuitively understood by students. The idea of adding or subtracting zero leaving the original number unchanged is relatively easily understood. However this arithmetic property requires exploration time because of it's later application in solving algebraic equations.
Students make use of the additive identity when faced with calculations like:
34 + 65 - 34 =
Finding the zero (n - n = 0) where n stands for any number means no calculation is required. The answer is simply 65.

In a simple algebraic equation x + 5 = 7 The rule of taking the five to the other side of the equal sign and reversing the operation is explained by applying the additive identity property.

x + 5 - 5 = 7 - 5
x = 2

One (1) is the Identity for Multiplication

It is the only number which does not alter the value of any number by which it is multiplied.

The multiplicative identity can be expressed in four different ways and each way written in two equivalent forms.

n x 1 = n or n = n x 1
1 x n = n or n = 1 x n
n ÷ 1 = n or n = n ÷ 1
n ÷ n = 1 or 1 = n ÷ n

Students will often incorrectly apply the additive identity property to a multiplication and think 5 x 0 = 5 or 0 x 5 = 5.

The multiplicative identity property needs to be explored as students generalise the commutative property. This property has applications in the understanding of place value. In accepting multiplication is about equal grouping, the column heading “ones” requires thinking of 5 groups of 1 and understanding 5 groups of 1 is the same as 1 group of 5.
This arithmetic property requires exploration because of it’s later use in solving algebraic equations.

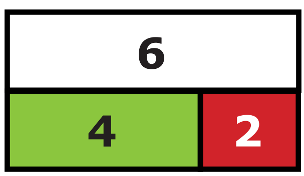
Student make use of the multiplicative identity property when faced with calculations like 46 x 28 ÷ 46
Finding the one (n ÷ n) where n stands for any number means no calculation is required. The answer is simply 28.

In a simple algebraic equation 6y = 42. The rule of taking the six to the other side and reversing the operation is explained by applying the multiplicative identity property.
6y ÷ 6 = 42 ÷ 6
y = 7

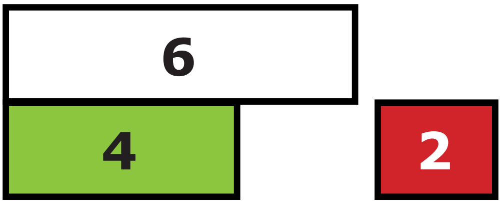
Inverse Relationships

Subtraction is the inverse of addition

Early experiences with subtraction are most often in the form of ”take away” and do not appear to have any relationship to addition situations. Subtraction is only fully understood when it can be seen as an aspect of addition.
This can be illustrated using number strips:



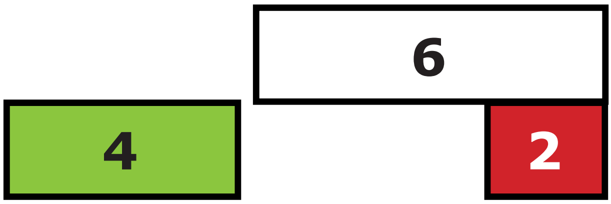
4 + 2 = 6   2 + 4 = 6



6 take away 2 = 4 6 – 2 = 4

4 and how many more make 6 4 + ? = 6

The difference between 6 and 4 is 2 6 - 4 = 2



6 take away 4 = 2 6 – 4 = 2

2 and how many more make 6 2 + ? = 6

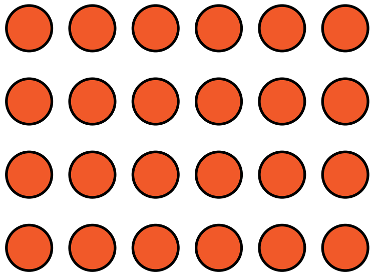
The difference between 6 and 2 is 4 6 – 2 = 4

Students need to come to an understanding of the relationships between addition and the different sorts of subtraction to fully understand the family of facts and begin to generalise the properties of addition and subtraction.

Division is the inverse of multiplication.

The use of arrays from early in a student's experience of multiplication helps to illustrate the relationship between multiplication and division.

Students need to come to an understanding of the relationships between multiplication and division to fully understand the family of facts and begin to generalise the properties of multiplication and division.



Can you find all of the following statements in the above array?

4 rows of 6 = 24
6 rows of 4 = 24
24 divided into 4 rows = 6 in each row
24 divided into 6 rows = 4 in each row
24 grouped in sixes = 4 groups
24 grouped in fours = 6 groups
A quarter of 24 = 6
A sixth of 24 = 4
$4 \times 6 = 24$ $\quad 6 \times 4 = 24$
$6 \times ? = 24$ $\quad 4 \times ? = 24$
$24 \div 6 = 4$ $\quad 24 \div 4 = 6$

Understanding the properties of arithmetic lays the foundation for students to understand algebra as generalised arithmetic.